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Mathematical formulation of the laws of conservation of mass and energy and the equation of motion for a moving thread

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SUMMARY
The general mathematical formulation of the laws of conservation of mass and energy and the equation of motion are derived for a moving thread. Within the thread only forces in tangential direction are considered: internal stresses perpendicular to the tangential direction are left out of consideration. The equations as formulated here can easily be applied to many problems in the various stages of fibre processing.

1. Introduction

In order to derive a mathematical formulation for the behaviour and the properties of a running thread that passes thread guides, speed regulators, friction elements, heating elements etc. three fundamental physical laws are essential. They are the law of conservation of mass, the equation of motion and the law of conservation of energy (first law of thermodynamics). In this paper the general mathematical formulation of these laws is derived for a running thread. Within the thread only tangential forces are taken into consideration. Besides, all mechanical, structural and heat properties are assumed to be the same in all points of any cross section of the thread. So the behaviour and properties of a particle within the thread are fully characterized by the cross section to which the particle belongs and by time*. Furthermore, throughout this paper it is assumed that no mass is supplied to or withdrawn from the thread.

The physical laws will be formulated for a part of the yarn with a finite length in the form of integral equations. From them partial differential equations are derived. By following a moving particle the equations are given in Lagrangian coordinates; with respect to a fixed system of coordinates the equations are given in Eulerian coordinates. Both systems of coordinates are taken into consideration.

The reader who is interested in this material may be referred to the well-known book by Prager [9], which deals with particles that move with respect to a fixed system of coordinates (e.g. water in a vessel); this paper discusses a one-dimensional extension of this theory to particles which move within a system (the thread) that is in motion itself.

Several authors deal with special problems in this field. E.g., ballooning is studied by Mack [7], Gerdes and De Maat [5] and Ames, Lee and Zaiser [1]; yarn twisting on rollers by Thwaites [10]; spinning is dealt with by Kase and Matsuo [6], and Pearson and Matovich [8]; non-linear vibration of running threads is considered by Ames, Lee and Zaiser [1], see also Broer [2].

The purpose of this general study was to increase the insight into concepts, definitions and mathematical formulation of the physical phenomena during the textile processes of synthetic

* So, torsion around the thread axis is not taken into consideration.

yarns. The properties of the yarn at the end of the textile process also depend on rheology, crystallinity, orientation etc. of the synthetic product. These properties are mostly described by empirical expressions based on the results of lab-scale experiments. These expressions are not dealt with in this paper. It is indicated, however, where these relations are of importance in the mass, momentum and energy balances. The general form of the equations enables one to apply them easily to many problems in various stages of fibre processing, such as spinning, drawing, twisting on rollers and ballooning on spindles.

2. Definition and concepts

We choose a fixed system of coordinates X, Y, Z in the three-dimensional space with origin O. In this space moves a thread. An *arbitrary* point O′ of the thread is chosen as a reference point. O′ may be fixed on the thread (e.g. an end), or moving along the thread. O′ may also be fixed or moving in space (e.g. a thread guide).

Each cross section of the thread will be considered as a mathematical point. We call such a point a particle or yarn element. In order to characterize the position of any particle along the thread the following points of view are possible:

a) *Eulerian Coordinates*
The position of any element is defined by the arc length s, i.e. the actual length measured along the thread from reference point O′ at time t, i.e. real time measured from an initial time t_0.

The position $\boldsymbol{r}$ of any thread element (see Fig. 1) with respect to the coordinates X, Y, Z is a function of s and t.

$$\boldsymbol{r}(s, t) = (x(s, t)\,,\; y(s, t)\,,\; z(s, t))\,. \tag{1}$$

If t is constant, this function $\boldsymbol{v} = \boldsymbol{v}(s)$ gives the actual position of the thread in space at time t. If s is constant, $\boldsymbol{r} = \boldsymbol{r}(t)$ describes the movement in time of the particles at distance s from O′.

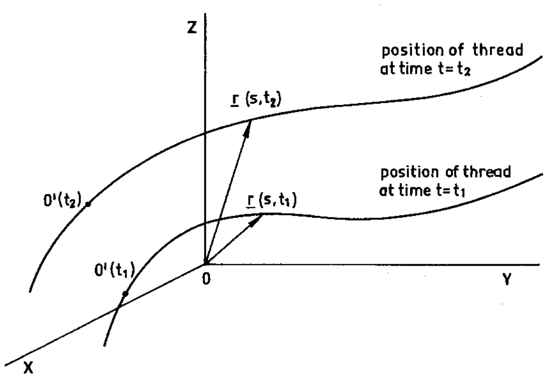

Figure 1. The thread moves within a three-dimensional space.

b) *Lagrangian Coordinates*
In this case, any thread element is labelled by l, which is defined as the arc length with respect to reference point O′ at initial time t_0.

In order to avoid confusion time is here denoted by τ. In fact $t \equiv \tau$. The position $\boldsymbol{r}$ of any thread element is a function of l and τ:

$$\boldsymbol{r}(l, \tau) = (x(l, \tau)\,,\; y(l, \tau)\,,\; z(l, \tau))\,.$$

If τ is constant, $\boldsymbol{r} = \boldsymbol{r}(l)$ describes the thread position in space. If l is constant, $\boldsymbol{r} = \boldsymbol{r}(\tau)$ describes the movement of thread element l in time.

With these definitions, the relations between the coordinates s and t (Eulerian coordinates) and the coordinates l and τ (Lagrangian coordinates) are given by:

$$t=\tau\,,\quad s=s(l,\tau)\,. \tag{2}$$

We define:

$$s_\tau = v \tag{3}$$

and

$$s_l = 1+\varepsilon \tag{4}$$

v is by definition the arc length passed per unit time by the particle. ε is the strain or elongation. The strain at initial time t_0 is assumed to be 0. ε and v are functions of s and t, or l and τ respectively. In the literature the symbols Df/Dt and $\dot{f}$ are also used for f_τ.

The velocity of a particle referring to the original system of coordinates X, Y and Z is given by:

$$\boldsymbol{w} = \boldsymbol{r}_\tau \tag{5}$$

or

$$\boldsymbol{w} = \boldsymbol{r}_t + v\boldsymbol{r}_s\,. \tag{6}$$

To the velocity $\boldsymbol{w}$ of a particle contribute $v\boldsymbol{r}_s$ as the velocity of the particle along the thread, and the term $\boldsymbol{r}_t$ as the movement of the thread in position s. In stationary situations, v is the yarn speed and equals the length of $\boldsymbol{w}$.

By definition of the arc length s:

$$\boldsymbol{r}_s \cdot \boldsymbol{r}_s = 1\,. \tag{7}$$

Two useful properties can be derived from (7) in a simple way. Differentiation of (7) with respect to s yields:

$$\boldsymbol{r}_{ss} \cdot \boldsymbol{r}_s = 0 \tag{8}$$

and to t:

$$\boldsymbol{r}_{st} \cdot \boldsymbol{r}_s = 0\,. \tag{9}$$

So, the vectors $\boldsymbol{r}_{ss}$ and $\boldsymbol{r}_{st}$ lie in the plane perpendicular to the unit tangent $\boldsymbol{r}_s$.

$$\boldsymbol{r}_{ss} = \frac{1}{R}\,\boldsymbol{n} \tag{10}$$

where R is the radius of curvature and $\boldsymbol{n}$ is the main normal with unit length.

Whether we will formulate and solve our problem in Eulerian coordinates (s and t) or in Lagrangian coordinates (l and τ) depends on the complexity of the equations and of the boundary conditions. Generally, the system of differential equations is simpler in Lagrangian coordinates than in Eulerian coordinates, because the law of conservation of mass, the equation of motion and the law of conservation of energy are all related to the moving element. Also other physical and structural properties are often connected with the particle in motion. However, the apparatus influencing the movement of the thread is mostly given in space. The initial and boundary conditions have therefore in the Eulerian coordinates mostly a simpler form than in Lagrangian coordinates. In this paper the physical laws will be formulated in both systems of coordinates.

It should be observed that the definition of the meaning of the Lagrangian coordinates l and τ of this paper is not the only possible one. For instance, the meaning of time τ may also be defined as the "residence" time of a particle, from the moment that this passes the guide point O′. With this definition of τ equation (2) has to be extended.

In this way the boundary conditions can sometimes be simplified. In this paper the meaning of τ will be restricted, however, to the real time measured from the initial time t_0.

At the end of this chapter two important properties are mentioned that will be used several times in the next chapters.

a) If $\int_s f\,ds = 0$ for any length s, then $f=0$ (11)

b) For a function $f=f(s,t)=f(l,\tau)$ differentiable both with respect to s and t and with respect to l and τ, the following relations hold:

$$\frac{\partial}{\partial\tau}\int_s f ds = \frac{\partial}{\partial\tau}\int_l f s_l dl = \int_l (f s_l)_\tau dl = \int_s (f s_l)_\tau s_l^{-1} ds \tag{12}$$

or, with (3):

$$\frac{\partial}{\partial\tau}\int_s f ds = \int_s (f_\tau + f v_s) ds = \int_s \{f_t + (f v)_s\} ds . \tag{13}$$

Eq. (13) is sometimes called Reynolds' transport theorem, see e.g. [4, p. 282].

3. Law of conservation of mass

Let ρ be the density of the material, ρ' the linear density or mass per unit length and A the area of a cross-section. The following relation holds: $\rho'=\rho A$. A, ρ and ρ' are functions of l and τ, or s and t.

The law of conservation of mass states that the change of mass of a certain finite thread element per unit time is zero, since no mass is supplied to or withdrawn from the thread. It can be formulated as follows.

$$\frac{\partial}{\partial\tau}\int_s \rho' ds = 0 \tag{14}$$

s is the actual arc length of the thread element considered, and depends on τ. Using (11) and (12), we can also write

$$(\rho' s_l)_\tau = 0 . \tag{15}$$

So, using (4):

$$\rho'(1+\varepsilon) = \text{a function of } l . \tag{16}$$

In other words, $\rho'(1+\varepsilon)$ is constant for each particle. In the special case that at $t=t_0$ the thread is homogeneous, which means that the density and area of cross section are the same in all points of the thread, we may write:

$$\rho'(1+\varepsilon) = \text{constant along the whole thread} . \tag{17}$$

With respect to the Eulerian coordinates s and t and using (11), (13) and (14) the mass balance can be written as follows:

$$\rho'_t + (\rho' v)_s = 0 . \tag{18}$$

4. Equation of motion

The forces acting on a segment of a moving thread with a given length s are represented in Figure 2.

a) The yarn load at the ends of the thread segment:

$$\boldsymbol{F}(s_2) - \boldsymbol{F}(s_1) = \int_s \boldsymbol{F}_s ds .$$

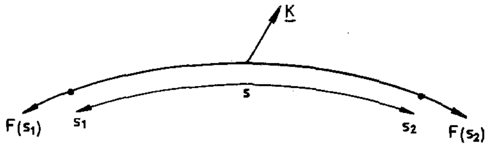

Figure 2. Forces acting on a thread segment s.

Only tangential forces are taken into consideration:

$$\boldsymbol{F} = F\boldsymbol{r}_s\,. \tag{19}$$

b) The external force $\boldsymbol{K}$ acting on the segment s can be distinguished into a contribution of volume or mass forces such as gravity, and the contribution of external forces acting on the surface of the thread such as a friction. If the behaviour of the thread was formulated in more dimensions, these latter forces would determine the boundary conditions. In our cases, however, where the thread is considered one-dimensional, these external forces can also be dealt with as volume forces. Introducing $\boldsymbol{K}'$ as the force acting per unit of actual length, we may write:

$$\boldsymbol{K} = \int_s \boldsymbol{K}'\,ds\,. \tag{20}$$

The equation of motion states that the change of momentum of a yarn element per unit time is equal to the total force acting on the element. Using (19) and (20), it can be formulated as:

$$\frac{\partial}{\partial\tau}\int_s \rho'\boldsymbol{w}\,ds = \int_s (\boldsymbol{F}_s+\boldsymbol{K}')\,ds\,.$$

With (11) and (12) and the mass balance (15), we may write:

$$\rho'\boldsymbol{w}_\tau = \boldsymbol{F}_s+\boldsymbol{K}'\,. \tag{21}$$

Using (6) and (19) this equation can be written with respect to the Eulerian coordinates s and t as follows:

$$\rho'\boldsymbol{r}_{tt}+\rho'\boldsymbol{r}_s(v_t+vv_s)+\rho'v^2\boldsymbol{r}_{ss}+2\rho'v\boldsymbol{r}_{st} = F_s\boldsymbol{r}_s+F\boldsymbol{r}_{ss}+\boldsymbol{K}'\,. \tag{22}$$

The terms on the left-hand side of (22) have the following meaning:

(i)	$\rho'\boldsymbol{r}_{tt}$	acceleration of the thread for a fixed coordinate s
(ii)	$\rho'(v_t+vv_s)=\rho's_{\tau\tau}$	acceleration of a particle along the threadline,
(iii)	$\rho'v^2\boldsymbol{r}_{ss}=\rho'v^2\boldsymbol{n}/R$	centrifugal force, perpendicular to the thread, see (8) and (10),
(iv)	$2\rho'v\boldsymbol{r}_{st}$	Coriolis' force, perpendicular to the thread, see (9).

By multiplying (22) with $\boldsymbol{r}_s$ and using (7), (8) and (9) the equation of motion in tangential direction can be written as:

$$\rho'\boldsymbol{r}_{tt}\cdot\boldsymbol{r}_s+\rho'(v_t+vv_s) = F_s+\boldsymbol{K}'\cdot\boldsymbol{r}_s\,. \tag{23}$$

5. Law of conservation of energy

If u is defined as the internal energy per unit mass, $\rho'u$ is the internal energy per unit of actual length. We introduce the heat transferred to the thread per unit time and per unit of actual length: q'. The conductive heat flows through the ends s_1 and s_2 of a line segment s in positive direction are called $Q(s_1)$ and $Q(s_2)$.

The law of conservation of energy states that the sum of the total amount of heat transferred to the thread segment s per unit time and the total work done by forces acting on the thread segment s per unit time is equal to the change of the sum of the internal energy and kinetic energy of the particles of the thread segment per unit time.

In formula this leads to:

$$\frac{\partial}{\partial\tau}\int_s (\rho'u+\tfrac{1}{2}\rho'\boldsymbol{w}\cdot\boldsymbol{w})\,ds = \int_s q'\,ds-Q(s_2)+Q(s_1)+\boldsymbol{F}\cdot\boldsymbol{w}(s_2)-\boldsymbol{F}\cdot\boldsymbol{w}(s_1)+\int_s \boldsymbol{K}'\cdot\boldsymbol{w}\,ds\,.$$

Using the mass balance (15), it follows from (11) and (13) that:

$$\rho'(u+\tfrac{1}{2}\boldsymbol{w}\cdot\boldsymbol{w})_\tau = q'-Q_s+(\boldsymbol{F}\cdot\boldsymbol{w})_s+\boldsymbol{K}'\cdot\boldsymbol{w}\,.$$

From the equation of motion (21) it follows that:

$$\rho' \boldsymbol{w} \cdot \boldsymbol{w}_\tau = \boldsymbol{w} \cdot \boldsymbol{F}_s + \boldsymbol{K}' \cdot \boldsymbol{w} .$$

So, the energy equation reads as follows:

$$\rho' u_\tau = q' - Q_s + \boldsymbol{F} \cdot \boldsymbol{w}_s . \tag{24}$$

Using (19) and (6) through (9), the third term on the left-hand side of (24), which is the deformation work per unit of actual length, can be written as follows:

$$\boldsymbol{F} \cdot \boldsymbol{w}_s = F \boldsymbol{r}_s \cdot (\boldsymbol{r}_{st} + v_s \boldsymbol{r}_s + v \boldsymbol{r}_{ss}) = F v_s . \tag{25}$$

So, the energy balance may be written as:

$$\rho' u_\tau = q' - Q_s + F v_s . \tag{26}$$

In fact, this equation is the first law of thermodynamics: the change in internal energy equals the sum of heat and work supplied. The equation (26) has been written in general terms. For synthetic yarns the internal energy u will generally be an intricate function of temperature, crystallinity, orientation of the molecules, entropy etc. If we are interested in both the temperature of the thread and all the yarn properties, the relations between these properties and the temperature, elongation and tension have to be known. Besides the mass balance, equation of motion and energy balance, also the rheology, equation of state and possible other relations have to be known then. In this paper no attention is given to these important relations. At the end of this chapter a more practical form for the energy balance will be given for a situation where the internal energy only depends on the temperature T and the crystallinity α of the yarn:

$$u = u(T, \alpha) .$$

Moreover, it is assumed that heat transfer due to radiation and convection contributes to q' in such a way that the flow of heat through the surface area per unit length Ω' is proportional to the difference between the temperature T of the thread and the temperature T_0 of the surrounding medium:

$$q' = k\Omega' (T_0 - T)$$

where k is the coefficient of the heat transfer.

Assume that Fick's law holds:

$$Q = -\lambda A T_s$$

where λ is the heat conductivity and A the area of a cross section.

In this situation, (26) reads as follows:

$$\rho' c T_\tau - \rho' c^* \alpha_\tau = k\Omega' (T_0 - T) + (\lambda A T_s)_s + F v_s \tag{27}$$

where $c = u_T$ and $c^* = -u_\alpha$.

So, c is the specific heat at a constant crystallinity and c^* is the heat of crystallization at a constant temperature. In this example no elastic deformation has been taken into consideration. If the elastic deformation is important, the potential energy contributes to the internal energy u.

6. Example

An illustration of the formulation of the equation of motion is given by the steady-state balloon curve calculation for ring spindles, where air drag is taken into consideration, see [5], [7].

The problem is to describe the course of the yarn when the yarn is led through a guide point and a traveller ring, the latter rotating with a constant angular velocity ω around a vertical axis (see Fig. 3).

Through the fixed guide point O a system of coordinates X, Y, Z is chosen. The external force $\boldsymbol{K}'$ working on the thread is the air drag.

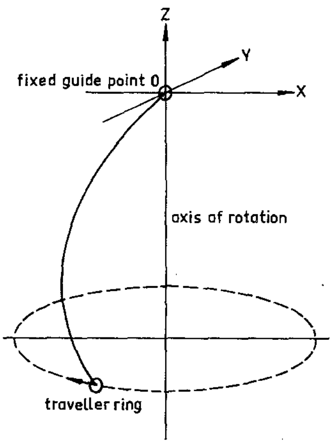

Figure 3. Balloon curve.

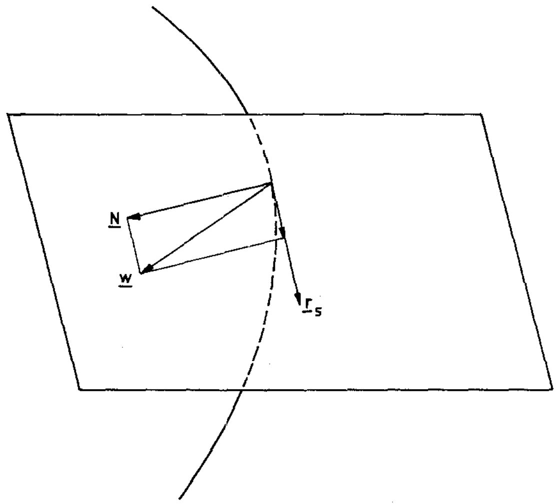

Figure 4. The direction N of the air drag.

It can be argued that only the component perpendicular to the thread should be taken into account, see [5], [7] and the references given there. So, the air drag is directed along the component $\boldsymbol{N}$ of the velocity $\boldsymbol{w}$ in the plane through $\boldsymbol{w}$ and the tangent $\boldsymbol{r}_s$; this component $\boldsymbol{N}$ is normal to the tangent (see Fig. 4).

The magnitude of the air drag per unit length of the thread is assumed to be cN^2. The coefficient c is given in the literature, see [5] for references.

So we may write:

$$\boldsymbol{K}' = -cN\boldsymbol{N} . \tag{28}$$

The vector $\boldsymbol{N}$ is given by:

$$\boldsymbol{N} = \boldsymbol{w} - (\boldsymbol{w} \cdot \boldsymbol{r}_s)\boldsymbol{r}_s$$

or, with (6) and (7):

$$\boldsymbol{N} = \boldsymbol{r}_t - (\boldsymbol{r}_t \cdot \boldsymbol{r}_s)\boldsymbol{r}_s . \tag{29}$$

Using (7) and (29), the following relation can be derived for N:

$$N^2 = \boldsymbol{r}_t \cdot \boldsymbol{r}_t - (\boldsymbol{r}_t \cdot \boldsymbol{r}_s)^2 . \tag{30}$$

The thread is assumed to be rigid; no strain can occur. Applying the mass balance as formulated in (17) and (18), it follows that v is constant along the thread.

We now introduce the following transformation of coordinates:

$$x = X(s) \cos \omega t - Y(s) \sin \omega t$$
$$y = X(s) \sin \omega t + Y(s) \cos \omega t$$
$$z = Z(s) .$$

So, we consider the stationary course of the thread referring to a system of coordinates X, Y, Z rotating with constant angular velocity around the z-axis.

Substitution of these transformation formulas in (22) and (28) through (30) leads after some simple calculation to the following equation:

$$-\rho' \omega^2 X - 2\rho' v\omega Y_s + \rho' v^2 X_{ss} =$$
$$= c\omega^2 \{X^2 + Y^2 - (XY_s - YX_s)^2\}^{\frac{1}{2}} \{Y + (XY_s - YX_s)X_s\} + (FX_s)_s .$$

In the same way the other equations for Y and Z are derived. In [5] and [7] boundary conditions and numerical solutions are given in detail.

7. Discussion

The mathematical formulation of the laws of conservation as given by the equations (7), (18), (22) and (26) do not suffice to describe a running thread completely: the yarn rheology and a description of the internal energy of yarn are needed, but this lies outside the scope of this paper.

However, it may be pointed out that the yarn rheology has some mathematical consequences that are important in solving the equations. We do not go into detail, but shall only indicate the line of argument.

Let us consider the following types of rheology.

(i) No deformation occurs, i.e. the strain ε is constant. For a homogeneous thread ρ' is constant, too, see (17). For an inhomogeneous thread ρ' follows from (16).

(ii) Stress depends only on strain or, aequivalently, on ρ', e.g. Hookean elasticity.

(iii) The third type of rheology is more complicated. Time effects are included. A relation exists between the local strain rate, the local strain and the local stress.

Assume that yarn temperature is constant, so that the energy balance need not be considered. The equations (7), (18), (22) in combination with a rheological description define the course of the running thread.

By deriving the characteristics, see e.g. [3, p. 425] it can be shown that the system is parabolic for the rheologies (i) and (iii), but hyperbolic for rheology (ii).

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